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MICROWAVE INVESTIGATION OF THE DIELECTRIC WAVEGUIDE PROPAGATION BY MAGNETO-IONIC DUCTS

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SUMMARY

The work performed under this Contract during the indicated two years period is described. Theoretical work includes : completion of the basic theory for propagation along plasma columns in magnetic fields, derivation of Brillouin diagrams, discussion of particularly significant limits and preliminary analyses of the non-uniform plasma case. Experimental work includes measurements of basic propagation parameters : transmitted signal, wavelength and group velocity, and of physical plasma parameters as the electron density.

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INTRODUCTION .

During the two years covered by the present report extensive theoretical and experimental work has been performed on the propagation of electromagnetic waves along plasma columns in longitudinal magnetic fields.

In the theory the most substantial contributions to the understanding of the propagation characteristics are the discussion of the dispersion equation solutions in the $\omega_b < \omega$ region, the evaluation of the power ratios in the same region and the derivation of the Brillouin diagrams. This analysis is reported in Technical Scientific Note N. 2. A more complete discussion of the propagation characteristics of circularly symmetrical modes along an uniform plasma column of circular cross-section will appear on the September issue of the Journal of Research, edited by the National Bureau of Standards.

In the present report we describe other theoretical considerations, which are of importance for understanding the propagation theoretical and experimental results.

The Brillouin diagrams in the limits of a very thin plasma ($d \to o$) and for a large cross-section plasma ($d \to \infty$) are discussed. By means of a Doppler shift they become the corresponding diagrams for an electron beam; it is interesting to rediscover in this way the well-known space-charge, cyclotron and syncronous waves.

As another special case, the Brillouin diagrams are discussed, when the following typical ionospheric conditions for whistler propagation are satisfied :

$$\omega^2 < \omega_b^2 \ll \omega_p^2$$

Purpose of it is to find whether the delay versus frequency characteristic of whistler atmospherics may be explained also assuming propagation

along well-defined plasma columns.

All the previous theoretical analyses assume an uniform density plasma, an assumption which is not satisfied in the experiments. The necessity to explain the observed experimental data has led us to consider also theoretically the effects of disuniformities.Results have been obtained on simplified aspects of this problem; they are reported in this document.

A very long report would be necessary to describe fully all the experiments and tentatives performed during these two years. Many technical difficulties have been encountered, due to the RF high-power involved in producing the discharge (gas purity and transfer of power are the most important), to the difficulty of exciting selectively the various propagation modes, to the necessary limitations in the dimensions of the experiment (which would have to be the scaled laboratory model of an ionospheric duct) and finally to the complexity of some of the measuring electronic equipment.

These experiments can be grouped in the following classes :

a) Transmission measurements. Purpose of these measurements was to collect more data over a wider range of experimental conditions, in order to confirm the previous experimental results and the general correctness of their qualitative interpretation.

b) Propagation wavelenght measurements. Previous data were insignificant, due to the presence at the same time of various modes and of undesired stray fields.

c) Electron density measurements. This is a basic quantity, which is necessary to know in order to compare propagation experiments with theory. Conventional methods had failed in our geometrical and physical conditions; new special methods had then to be used.

d) Group velocity measurements. From these data, knowing electron density and the other physical parameters, we can derive experimentally Brillouin diagrams. The same could be obtained in a more simple way

from the propagation wavelength measurements, but , as it is later explained in this report, we can measure these wavelengths only for the first mode, which is the less interesting one.

In the report detailed information is given on these experiments and on their results. In all the experiments the plasma has been produced using a new RF generator capable of delivering up to 4 KW of power at frequencies between 1.5 and 3 Mc.When specific data of the operating conditions will not be given, reference has to be made to the analogous cases reported in Technical Summary Report N.1.

COMPARISON WITH BEAM WAVES

It is enlightening to relate our propagation modes to the well-known space-charge, cyclotron and synchronous waves, which can be excited in electron beams.

This relation is best shown on a λ_g/λ_0 versus d/λ_g diagram, ε_1 and ε_3 being the fix parameters for each curve. These curves can be simply derived from the λ_g/λ_0 versus d/λ_0 plots, previously obtained from the theoretical analysis. The usual beam waves are derived in the microwave tube theory under the simplifying assumption of a phase velocity much smaller than the light velocity, or $\lambda_g/\lambda_0 \ll 1$, which is equivalent to our quasi-static approximation.

The result is that all the λ_g/λ_0 curves, near to the d/λ_g axis, which is the region where the quasi-static approximation holds, are vertical and given by the relation :

$$d/\lambda_g = \nu_\infty^{(m)}/\pi \tag{1}$$

where $\nu_\infty^{(m)}$ is the m - th root of the modified characteristic equation:

$$\frac{\varepsilon_3}{\sqrt{-\varepsilon_3/\varepsilon_1}} \; \frac{J_1\left(\nu_\infty^{(m)}\sqrt{-\varepsilon_3/\varepsilon_1}\right)}{J_0\left(\nu_\infty^{(m)}\sqrt{-\varepsilon_3/\varepsilon_1}\right)} = - \frac{K_1\left(\nu_\infty^{(m)}\right)}{K_0\left(\nu_\infty^{(m)}\right)} \tag{2}$$

The above λ_g/λ_0 vertical curves exist only when :

a) $1 \geq \varepsilon_3 \geq 0$, $\varepsilon_1 \leq 0$ (an infinite set of curves exists)

b) $\varepsilon_3 \leq 0$, $\varepsilon_1 \geq 1$ (an infinite set of curves exists)

c) $\varepsilon_3 \leq 0$, $\varepsilon_1 \leq 1/\varepsilon_3$ (only one curve exists m = 1)

In an infinitely extended plasma $d/\lambda_g \to \infty$. Then $\nu_\infty^{(m)}$ approaches infinity and the second member of the characteristic equation becomes equal to - 1.

Let us consider first the cases a) and b) above. The two dielectric components ε_1 and ε_3 have opposite signs and the equation becomes :

$$\lim_{\nu_m^{(n)}\to\infty} J_0\left(\nu_m^{(n)}\sqrt{-\varepsilon_3/\varepsilon_1}\right) = \frac{-\varepsilon_3}{\sqrt{-\varepsilon_3/\varepsilon_1}} \lim_{\nu_m^{(n)}\to\infty} J_1\left(\nu_m^{(n)}\sqrt{-\varepsilon_3/\varepsilon_1}\right) \qquad (3)$$

Two cases exist, in which this equation is satisfied :
$\varepsilon_3 \to 0$ as $-\varepsilon_1\left(\chi_0^{(m)}/\nu_m^{(n)}\right)^2$, and $\varepsilon_1 \to \infty$ as $-\varepsilon_3\left(\nu_m^{(n)}/\chi_1^{(m)}\right)^2$, where $\chi_n^{(m)}$ is the m - th root of the Bessel function J_n.

When ε_1 and ε_3 are both negative, I_0 and I_1 functions replace J_0 and J_1 and another possibility exists. This is $\varepsilon_1\varepsilon_3 = 1$, a condition which satisfies the characteristic equation, because the function ratio I_1/I_0 equals unity when its argument becomes infinity.

In a stationary plasma, $\varepsilon_3 = 0$ when $\omega = \omega_p$, $\varepsilon_1 \to \infty$ when $\omega = \omega_l$ and $\varepsilon_1\varepsilon_3 = 1$ when $\omega = \sqrt{(\omega_p^2 + \omega_l^2)/2}$, provided $\omega_l > \omega_p$. In an electron beam moving with velocity v_l along the magnetic field axis :

$$\varepsilon_1 = 1 - \frac{\omega_p^2}{(\omega - \beta v_l)^2 - \omega_l^2} \qquad (4)$$

$$\varepsilon_3 = 1 - \frac{\omega_p^2}{(\omega - \beta v_l)^2} \qquad (5)$$

Then $\varepsilon_3 = 0$ when :

$$\omega_p^2 = (\omega - \beta v_l)^2$$

$$\beta = (\omega \pm \omega_p)/v_l \qquad (6)$$

This is the well-known phase propagation characteristic of space charge waves.

The $\varepsilon_1 = \infty$ condition is satisfied when :

$$\left(\omega - \beta v_l\right)^2 - \omega_l^2 = 0$$

$$\beta = \left(\omega \pm \omega_l\right)/v_l \tag{7}$$

This is the phase propagation characteristic of cyclotron waves.
The $\varepsilon_1\varepsilon_3 = 1$ condition implies :

$$\left(\omega - \beta v_l\right)^2 - \frac{1}{2}\left(\omega_p^2 + \omega_l^2\right) = 0$$

$$\beta = \frac{\omega \pm \sqrt{(\omega_p^2 + \omega_l^2)/2}}{v_l} \tag{8}$$

Here too this case exists only if $\omega_l < \omega_p$. For this reason and because such waves are of the surface type, they are not considered in the usual microwave tube theory. They may however be of importance when a magnetic beam focusing is not used and when proper excitation is provided.

Let us consider now the other limit case $d/\lambda_g \to 0$. Then $\nu_\infty^{(n)}$ approaches zero and the K_1/K_0 ratio goes to infinity. One case, which satisfies this limit, is $\varepsilon_3 = -\infty$ (ε_1 being positive). Another set of solutions is given by the equation :

$$\lim_{\nu_\infty^{(n)} \to 0} J_0\left(\nu_\infty^{(n)} \sqrt{-\varepsilon_3/\varepsilon_1}\right) = 0 \tag{9}$$

This implies $\varepsilon_1 \to 0$, beside the previous $\varepsilon_3 \to -\infty$ case.

In a stationary plasma, $\varepsilon_3 = -\infty$ only when $\omega = 0$ and $\varepsilon_1 = 0$ when $\omega = \sqrt{\omega_p^2 + \omega_l^2}$.

In an electron beam, $\varepsilon_3 = -\infty$ when :

$$\omega - \beta v_l = 0$$

$$\beta = \omega / v_l \tag{10}$$

This is the phase propagation characteristic of synchronous waves.

In an electron beam, $\varepsilon_1 = 0$ when :

$$(\omega - \beta v_l)^2 - \omega_l^2 = \omega_p^2$$

$$\beta = \frac{\omega \pm \sqrt{\omega_p^2 + \omega_l^2}}{v_l} \tag{11}$$

In the theory of transverse - field microwave tubes these waves are also called cyclotron waves. In fact, we have usually $\omega_p^2 \ll \omega_l^2$, so that the phase characteristic at this limit coincides with the previously given characteristic for these waves in an infinitely extended beam.

It is interesting to see how the above picture of beam waves is modified, according to our analysis, when the complete theory is used in place of the results of the quasi-static approximation.

In the infinitely thin column (d $\to$ o) the complete theory does not contribute any further solution, beyond those given by the quasi-static analysis. In fact, the λ_g / λ_0 versus d/λ_0 curves show clearly that, when d $\to$ o, $\lambda_g / \lambda_0 \to 0$ always, so that this case implies always the validity of the quasi-static approximation.

The same λ_g/λ_o curves indicate instead more possibilities for the infinitely extended plasma. In fact, if $\varepsilon_1 \geq 1$ and $\varepsilon_3^{(o)} \leq \varepsilon_3 \leq 1$, all the investigated circularly symmetrical modes, except the first one, approach, when $d \to \infty$, the common ratio $\lambda_g/\lambda_o = 1/\alpha_o$. It is worth of mention that this same ratio is attained also in the case of a TEM plane uniform wave with a right hand polarization; however, fields are different and in our case we obtain :

$$\begin{cases} E_z = \text{const.} \\ E_\rho = j E_z \dfrac{\pi}{\lambda_o} \dfrac{\varepsilon_3}{\sqrt{\varepsilon_1 + \varepsilon_2}} \rho = j E_\varphi \\ H_z = \dfrac{j E_z}{Z_o} \dfrac{\varepsilon_3}{\sqrt{\varepsilon_1 + \varepsilon_2}} \\ H_\rho = -\dfrac{E_z}{Z_o} \dfrac{\pi}{\lambda_o} \varepsilon_3 \rho = j \dfrac{\sqrt{\varepsilon_1 + \varepsilon_2}}{Z_o} E_\rho = j H_\varphi \end{cases} \tag{12}$$

whereas in the TEM case one has :

$$\begin{cases} E_z = H_z = 0 \\ E_\rho = A/\rho = j E_\varphi \\ H_\rho = j \dfrac{\sqrt{\varepsilon_1 + \varepsilon_2}}{Z_o} E_\rho = j H_\varphi \end{cases} \tag{13}$$

When $\varepsilon_1 > 1$ and $\varepsilon_3 < \varepsilon_3^{(o)}$ the common λ_g/λ_o limit as $d \to \infty$ is $1/\alpha_i$, for all modes except the first one ; in this limit $A_2 = -A_1$ and the total field then vanishes in the infinite plasma.

The propagation characteristics for a plasma column or for a beam with an extremely large cross-section can be easily derived from the above considerations (the first mode will be neglected, due to its low capacity of carrying microwave power inside). In order to have $\varepsilon_1 \geq 1$, we assume $\omega < \omega_b$. Starting from $\omega \simeq 0$, ε_3 is a large negative number, certainly less than $\varepsilon_3^{(o)}$. In this region, substituting for ε_1 and ε_3

their values in the α_i formula, we obtain :

$$\beta = \frac{\omega}{c}\left\{1 + 2\,\frac{\omega_p^2-\omega^2}{\omega_b^2}\left[1+\sqrt{1+\frac{\omega_b^2}{\omega_p^2-\omega^2}}\,\right]\right\}^{1/2} \tag{14}$$

There will be a frequency at which this β becomes equal to the β value similarly derived from the α_o formula after substitution of the ε_1 and ε_3 values :

$$\beta = \frac{\omega}{c}\left\{1 + \frac{\omega_p^2}{\omega(\omega_b-\omega)}\right\}^{1/2} \tag{15}$$

For ω values larger than this one, the β versus ω curve follows the last formula up to $\omega = \omega_b$.

When $\omega_p < \omega_b$, as in most microwave tubes, the curve intersects the $\omega = \omega_p$ horizontal line, given previously by the quasi-static analysis; the diagram β versus ω for very large d values shows the continuity behaviour sketched in fig.1.

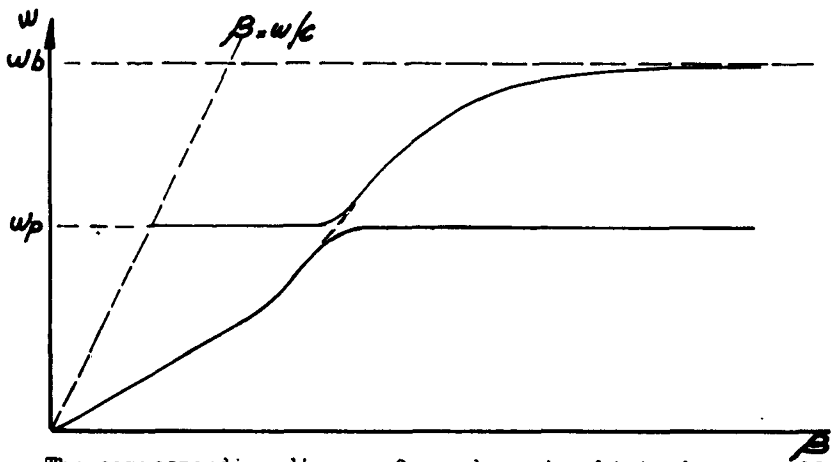

FIG 1

The corresponding diagram for a beam is obtained, as usually, by substituting the doppler shifted frequency $\omega - \beta v_b$ to signal frequency ω.

WHISTLER PROPAGATION

The possibility of finding a region, on the kd versus βd diagram, where the propagation shows the typical $\beta \sim \sqrt{\frac{\omega}{\omega_L - \omega}}$ behaviour of whistler propagation in an uniform ionosphere, has been reexamined on the basis of our most recent results for uniform cylindrical plasma columns.

We are assuming here the following conditions, which are satisfied along the ionospheric path of propagation of whistler atmospherics :

$$\omega^2 < \omega_L^2 \ll \omega_p^2 \qquad (16)$$

This means that ε_3 is negative, whereas ε_1 is positive; both, however, are much larger than unity.

Recalling the shape of the previously discussed $\alpha^{-1} = \lambda_g/\lambda_0$ versus kd or βd curves for constant ε_1 and ε_3, we see that each of them consists basically of two straight line parts. It is reasonable to expect that over any significantly wide frequency range the shape of the resulting kd versus βd curves are correctly predicted by the elaboration of one alone of the two sets of straight lines.

Let us consider first the constant α part of the (α^{-1} versus kd) curves, which is typical of the infinite plasma condition. We have $\alpha_L \sim 1$ for the first solution (ε_1 and $|\varepsilon_3| \gg 1$), α_o or α_i for all the other solutions. When ε_1 is very large :

$$\varepsilon_3^{(0)} \simeq -3\varepsilon_1 \qquad (17)$$

Then the condition : $\varepsilon_3 < \varepsilon_3^{(0)} = -3\,\varepsilon_1$, for which the constant α value

is α_i , is attained when :

$$\omega < \frac{\omega_b}{2} \tag{18}$$

In our case (ε_1 and $|\varepsilon_3| \gg 1$) :

$$\alpha_i^2 \simeq -4\,\frac{\varepsilon_1 \varepsilon_3}{\varepsilon_1 - \varepsilon_3} \simeq 4\,\frac{\omega_p^2}{\omega_b^2} \tag{19}$$

$$\alpha_0^2 = \varepsilon_1 + \varepsilon_2 \simeq \frac{\omega_p^2}{\omega(\omega_b - \omega)} \tag{20}$$

Correspondingly we have :

$$\beta \simeq \omega / c \qquad \text{first solution} \tag{21}$$

$$\beta \simeq \frac{2\omega\omega_p}{c\,\omega_b} \qquad \text{other solutions, when } \omega < \omega_b/2 \tag{22}$$

$$\beta \simeq \frac{\omega_p \sqrt{\omega}}{c\sqrt{\omega_b - \omega}} \qquad \text{other solutions, when } \omega_b/2 < \omega < \omega_b \tag{23}$$

The classical whistler propagation behaviour may thus be found also in plasma columns, but only over the frequency region between the cyclotron value and its half. Over this region whistler delay versus frequency curves show a characteristic rising behaviour.

Let us consider now the set of straight lines diverging from the origin in the α^{-1} versus ka plane (quasi-static approximation region).

For discussion simplicity it is convenient to solve this problem in the similar case of a plane geometry, where the basic equation of

these lines is :

$$tg\left(\beta b\sqrt{-\varepsilon_3/\varepsilon_1}\right) = 1/\sqrt{-\varepsilon_1\varepsilon_3} \ll 1 \qquad (24)$$

b being half the plasma thickness.
The solution is approximately :

$$\beta b\sqrt{-\varepsilon_3/\varepsilon_1} \simeq m\pi + 1/\sqrt{-\varepsilon_1\varepsilon_3} \qquad (25)$$

When $m = 0$:

$$\beta b = \frac{1}{|\varepsilon_3|} \simeq \frac{\omega^2}{\omega_p^2} \qquad (26)$$

When $m \neq 0$:

$$\beta b \simeq m\pi\sqrt{\frac{\varepsilon_1}{-\varepsilon_3}} = m\pi\frac{\omega}{\sqrt{\omega_p^2-\omega^2}} \qquad (27)$$

The whistler characteristic, which is found in uniform plasmas, is here never attained.

It seems then possible to conclude that in general propagation of circularly symmetrical fields along ionospheric plasma columns does not explain whistler typical characteristics. In fact, we may justify at most only the rising part of the whistler delay versus frequency curve near cyclotron resonance. Moreover, our theoretical curve must join, at frequencies lower than half the cyclotron value, a constant delay instead of showing the typical behaviour(frequency drops as the square of time delay).

PROPAGATION IN NON-UNIFORM COLUMNS

Various investigations on the changes of the propagation characteristics due to a non - uniform electron density in the plasma column have been performed for the purpose to explain some experimental results.

A general method for taking into account longitudinal disuniformities has been worked out by Prof. Cambi of the Rome University, on a consultant basis for this Contract. His analysis is reported in Appendix A.

More detailed analyses have been performed on the effects of transverse disuniformities. For analytical simplicity we have always considered a plane geometry, instead of our cylindrical one. In this case we have assumed density to be dependent only on the coordinate of the axis perpendicular to the plasma boundary.

As a simple case we have first investigated the propagation conditions when the electron density drops linearly from a maximum at the center to a finite value at the plasma boundary and when no magnetic field is present. This same case has been investigated and similar results reported at about the same time by Prof. Schumann.[1] For this reason and because of the scarce interest of this case for the interpretation of our experiments, no further discussion will be presented here.

Details will instead be given of a simple, but here more interesting case.

As it is well-known, in the quasi-static approximation retardation effects are neglected and the a.c. electric field is derived from a scalar potential:

$$\vec{E} = - \nabla \phi \qquad (28)$$

whereas a.c. magnetic fields are entirely neglected. We consider a slab of plasma in free space; a coordinate axis system is chosen as shown in figure 2. The constant magnetic field is directed along the positive z axis. Plasma electron density is a function of the transverse x coordinate only, and so are ε_1, ε_2 and ε_3.

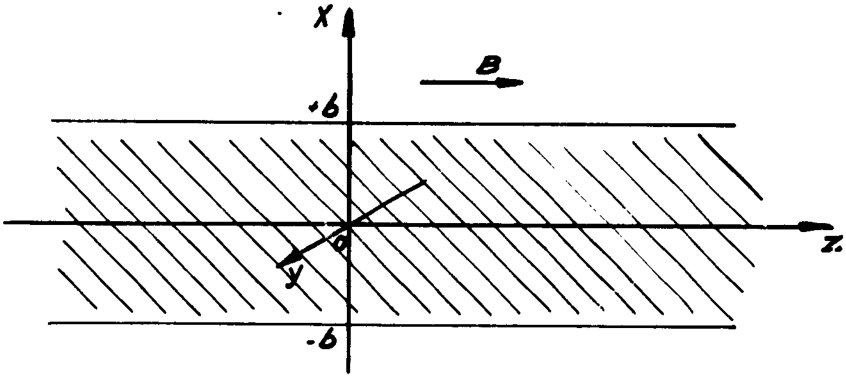

FIG. 2

Maxwell divergence equation provides then the following differential equation for ϕ :

$$\varepsilon_1 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + \varepsilon_3 \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial \varepsilon_1}{\partial x} \frac{\partial \phi}{\partial x} + j \frac{\partial \varepsilon_2}{\partial x} \frac{\partial \phi}{\partial y} = 0 \qquad (29)$$

In the plane geometry the modes, which correspond to the circularly symmetrical modes of the cylindrical geometry, require $\partial\phi/\partial y = 0$, and then the differential equation for ϕ becomes :

$$\varepsilon_1 \frac{\partial^2 \phi}{\partial x^2} + \varepsilon_3 \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial \varepsilon_1}{\partial x} \frac{\partial \phi}{\partial x} = 0 \qquad (30)$$

Let us assume now that typical whistler conditions (16) are

satisfied, so that :

$$\varepsilon_1 \simeq \frac{\omega_p^2}{\omega_b^2 - \omega^2} \tag{31}$$

$$\varepsilon_3 \simeq - \frac{\omega_p^2}{\omega^2} \tag{32}$$

Their ratio is thus density independent and is not function of position. Eq. (30) becomes :

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{|\varepsilon_3|}{\varepsilon_1} \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{\varepsilon_1} \frac{\partial \varepsilon_1}{\partial x} \frac{\partial \phi}{\partial x} = 0 \tag{33}$$

A simple solution is obtained if the density varies esponentially :

$$\varepsilon_1 = \varepsilon_1(0)\, e^{-ax} \tag{34}$$

We may call $1/a$ the characteristic length of the density disuniformity. Eq. (33) becomes :

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{|\varepsilon_3|}{\varepsilon_1} \frac{\partial^2 \phi}{\partial z^2} - a \frac{\partial \phi}{\partial x} = 0 \tag{35}$$

which has as a suitable solution :

$$\phi = A \cos \left[\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1} \beta^2 - \left(\frac{a}{2}\right)^2}\; x + \varphi \right] e^{ax/2}\, e^{-j\beta z} \tag{36}$$

For continuity $E_x(x = o)$ must vanish, and then $\partial\phi/\partial x\big|_{x=o} = 0$. This leads to the following relation for φ :

$$\mathrm{tg}\,\varphi = \frac{a/2}{\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}\beta^2 - \left(\frac{a}{2}\right)^2}} \tag{37}$$

Outside the plasma a suitable solution for the potential is :

$$\phi = B\,e^{-\beta x}\,e^{-j\beta z} \qquad (x > o) \tag{38}$$

At the plasma boundary (x = b) D_x and E_z must be continuous. This leads to two relations between A and B, which are satisfied only if the propagation constant β and the dielectric constant components are related by the following dispersion relation :

$$\mathrm{Tg}\left[\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}\beta^2 - \left(\frac{a}{2}\right)^2}\,b + \varphi\right] = \frac{\frac{\beta}{\varepsilon_1} + \frac{a}{2}}{\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}\beta^2 - \left(\frac{a}{2}\right)^2}} \tag{39}$$

where ε_1 is evaluated at the plasma boundary .
The second member of (39) can be rewritten as :

$$\frac{\beta}{\varepsilon_1\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}\beta^2 - \left(\frac{a}{2}\right)^2}} + \mathrm{tg}\,\varphi \tag{40}$$

The first of these two terms, as we shall show later, becomes negligible for all modes except the lowest one, and eq. (39) has then the following solutions :

$$\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}\beta^2 - \left(\frac{a}{2}\right)^2}\,b = m\pi \tag{41}$$

from which :

$$\beta b = \sqrt{\frac{\varepsilon_1}{|\varepsilon_3|}\left[(m\pi)^2 + \left(\frac{ab}{2}\right)^2\right]} = \frac{\omega}{\sqrt{\omega_b^2 - \omega^2}}\sqrt{(m\pi)^2 + \left(\frac{ab}{2}\right)^2} \qquad (42)$$

The first term of (40) is then :

$$\frac{1}{\sqrt{\varepsilon_1|\varepsilon_3|}}\,\frac{\sqrt{(m\pi)^2 + (ab/2)^2}}{m\pi}$$

which is obviously negligible in whistler conditions and provided $m \neq 0$.

Eq. (42) has to be compared with (27). The result is that in the presence of an exponential density decay, the propagation constant β given by the uniform plasma theory has to be multiplied by the factor $\sqrt{1 + (ab/2m\pi)^2}$. When the plasma thickness is a few times the characteristic length of the disuniformity the effect may thus be large, particularly for the lower m modes.

PROPAGATION AND WAVELENGTH MEASUREMENTS

The experimental results reported in Technical Summary Report N.1 can be explained in a much more clear way with reference to the Brillouin diagrams derived during this period and reported in Technical Scientific Note n.2.

We recall that the transmitted signal is detected by an internal probe and that propagation is observed only when $\omega_l > \omega$.

Reference to figs. 3 and 4 indicates clearly that when $\omega_l < \omega$ propagation may take place only :

a) In the first mode, when $\omega_p > \sqrt{2\omega^2 - \omega_l^2}$. This is a surface wave and as such carries very low microwave power along the central axial region of the plasma column. For this reason this mode is not detected by the probe.

b) In the upper branch modes, when $\sqrt{\omega^2 - \omega_l^2} < \omega_p < \omega$. In most cases these waves are backward, and then are not properly launched by the helical couplers. These waves may carry also forward power, but only at frequencies very near to cyclotron resonance; near this resonance, however, waves are very strongly attenuated by electron collisions. All these facts may explain,why such waves have never clearly detected in our experimental arrangement.

When $\omega_l > \omega$ propagation takes place :

a) when $\omega_p < \omega$, over a limited range of densities and provided the plasma diameter is sufficiently large ;

b) when $\omega_p > \omega$, always.

In Technical Summay Report n.1 we have given reasons to believe that experimentally observed propagations are related to conditions b) above, namely $\omega < \omega_l, \omega_p$.

One of the experiments, initiated during the first year of the Contract and continued during the successive months of 1960, had the purpose to measure wavelengths, when propagation was taking place along the plasma column.These wavelengths are determined from the standing wave patterns measured with a pick-up probe, which slides along the column very closely to the outside surface of the plasma container.
To obtain significant data it was immediately apparent the necessity to reduce the intensity of some undesired, stray fields, which were definitely found to be present outside the plasma.
The situation was improved by a better choice and arrangement of absorbing materials around the tube. These materials reduce propagation between input and output couplers, as well as spurious coaxial and waveguide modes. However, it was found that residual propagation due direct coupling and to dielectric modes along the container glass walls could be effectively reduced only feeding the microwave signal directly into the plasma, instead of injecting it through the glass walls by means of an external helical coupler, as done in the previous experiments.
Many different types of inside couplers were tested for this purpose; finally a plane spiral antenna of small dimensions was chosen for its good coupling and mode selection properties.

Using this couplers, properly placed absorbing materials and a new accurate driving mechanism for the probe carriage, it was possible to perform significant wavelength measurements. The result is that the measured wavelengths are only slightly less than the corresponding free space wavelengths.

This is in agreement with our theoretical analysis, provided the signal detected by the sliding external probe is propagating according to the first circularly symmetrical mode. Such a possibility is highly plausibile, because the first mode represents a surface wave; in this case

fields just outside the plasma are strong and may mask easily higher order mode fields.

The opposite situation has to be found inside the plasma, where the transmission detecting probe is placed. For this reason we believe that the observed transmitted signal characteristics are in most cases due to modes higher than the first, whereas the externally measured wavelengths refer to first mode propagation.

During the two years covered by this Report further propagation experiments have been performed. We shall mention here only two of them.

In one experiment, we have used our typical set-up but a different frequency range, namely the 1300 Mc/s range. Basically, we have observed the same phenomena as in the 5000 Mc/s range, the only difference being an higher number of transmission peaks. This is in agreement with the assumption that propagation is observed when $\omega < \omega_b, \omega_p$.

In a second experiment we have investigated propagation characteristics using different tube diameters. We have found that the detected signal power dependes on the tube diameter and that this signal practically disappears when the diameter is less than 4 cm. This experiment was performed in pure Neon and in Neon contamined with Mercury, but no significant difference was observed changing the gas. To explain, with our physical picture, the observed behaviour we must suppose that the electron density decreases with the tube diameter (for instance, this may be due to larger diffusion losses); so that when d $<$ 4 cm the condition $\omega_p > \omega$ is no longer satisfied.

ELECTRON DENSITY MEASUREMENTS

In order to proceed from the previous qualitative considerations into quantitative verifications of the theoretical analysis, it is necessary to know the values of all basic physical parameters which characterize the problem. Geometrical data, signal frequency and magnetic field intensity are easily measured according to standard techniques. Difficulties arise, when we try to measure electron densities with conventional methods; the failures of these tentatives have already been described in Technical Summary Report N.1.

A new approach was then chosen. The value of the average electron density in the discharge tube is derived from measurements of the guided wavelengths of microwave signals propagating along the plasma, when metallic walls are placed around the tube in a waveguide arrangement. Except for the presence of the external cylindrical metallic wall, concentrically placed near to the discharge tube, all the other geometrical and physical conditions are unchanged.

A.W. Trivelpiece and R.W. Gould [2], who first proposed this method, compared it with other microwave measuring techniques, as the cavity and the scattering methods, and found that it provides results in good agreement with those given by the other methods.

The measuring set-up consists of a cylindrical waveguide, 7,0 cm I.D., with a longitudinal slot. A screw-drived carriage, traveling along the guide, carries the probe. The cut-off frequency of the empty waveguide is at 2500 Mc/s; measurements have been taken at lower frequencies, so that propagation modes derived from conventional waveguide modes do not exist. At these frequencies and when the density becomes infinitely large, the field configuration and the propagation characteristics approach those of the classical TEM coaxial mode.

Measurements are conveniently plotted as Brillouin diagrams $\beta = 2\pi/\lambda_g$ versus $k = 2\pi/\lambda_o$. Comparing these curves with a set of

theoretically evaluated diagrams for various plasma frequencies the experimental densities are determined. Theory necessarily assumes uniform plasma, so that the described data must be regarded as average density values.

The solution of the propagation problem in our real geometry (plasma bounded by a glass tube, air and metallic waveguide) was first carried out and solved with the aid of the so-called quasi-static approximation and with the assumption of uniform plasma density. Whereas the experimental conditions of other authors were such that the results of the quasi - static approximation are sufficiently accurate, our computations have shown that this is not the case in our experiment, so that we have faced the necessity of more complete computations based on the rigorous characteristic equation of the plasma waveguide with metallic walls.

Experimental β versus k data fall on a straight line, passing through the origin; theoretically then, we must evaluate only the value of the Brillouin diagram slope $k/\beta = 1/\alpha$ at the limit $k = 0$.

Assuming no glass walls around the plasma, the dispersion equation is derived along the same lines used in the free space case discussed in Technical Scientific Notes n.1 and 2. For circularly symmetrical modes we obtain :

$$\tau \, \frac{\dfrac{1}{x_0} F' + \dfrac{1}{x_1} \dfrac{J_1(x_1)}{J_0(x_1)}}{\dfrac{1}{x_0} F'' + \dfrac{\varepsilon_3}{x_1} \dfrac{J_1(x_1)}{J_0(x_1)}} = \frac{\dfrac{1}{x_0} F' + \dfrac{1}{x_2} \dfrac{J_1(x_2)}{J_0(x_2)}}{\dfrac{1}{x_0} F'' + \dfrac{\varepsilon_3}{x_2} \dfrac{J_1(x_2)}{J_0(x_2)}} \qquad (43)$$

where

$$\begin{cases} F' = \left[\dfrac{K_1(x_0)}{K_1(x_0 b)} - \dfrac{I_1(x_0)}{I_1(x_0 b)} \right] \Big/ \left[\dfrac{K_0(x_0)}{K_1(x_0 b)} + \dfrac{I_0(x_0)}{I_1(x_0 b)} \right] \\ \\ F'' = \left[\dfrac{K_1(x_0)}{K_0(x_0 b)} + \dfrac{I_1(x_0)}{I_0(x_0 b)} \right] \Big/ \left[\dfrac{K_0(x_0)}{K_0(x_0 b)} - \dfrac{I_0(x_0)}{I_0(x_0 b)} \right] \end{cases} \qquad (44)$$

b = metallic waveguide to plasma diameter ratio

x_0, x_1, x_2, = see Technical (Scientific) Note n. 1

We are interested in the low frequency range of each Brillouin diagram, where the following approximations can be made :

$$\varepsilon_3 \simeq - \frac{\omega_p^2}{\omega^2} = - \frac{K_p^2 d^2}{K^2 d^2} \to - \infty \qquad (K_p = \omega_p / c)$$

$$x_0 = \frac{1}{2} K d \sqrt{\alpha^2 - 1} \to 0$$

Under these conditions, the above defined parameters become :

$$F' \simeq \frac{1}{2} x_0 \left(b^2 - 1 \right)$$

$$F'' \simeq \frac{1}{x_0} \left(\ln b \right)^{-1}$$

$$\frac{x_{1,2}^2}{K_p^2 d^2} \simeq \frac{(\alpha^2 - 1) - 2(\varepsilon_1 - 1) \pm \sqrt{(\alpha^2 - 1)^2 - 4 \alpha^2 (\varepsilon_1 - 1)}}{4 \varepsilon_1 \sqrt{\alpha^2 - 1}}$$

$$\tau = \frac{1 - \sqrt{1 - 4 \alpha^2 (\varepsilon_1 - 1)/(\alpha^2 - 1)^2}}{1 + \sqrt{1 - 4 \alpha^2 (\varepsilon_1 - 1)/(\alpha^2 - 1)^2}}$$

When these expressions are substituted into (43), the dispersion equation can be regarded as a relation between $k_p d$ and α^{-1} (which is now equal to the initial slope of the Brillouin diagram), provided we assume that $a = K_1^2 d^2 / K_p^2 d^2$ is a constant parameter ($k_b = c^{-1}\omega_l$). The α versus $k_p d$ curves can be found rather easily by graphical methods.

From these curves $k_p d = f_1(\alpha, a)$ and the equation $k_p d = K_1 d/\sqrt{a}$, a new set of curves $k_p d = f_2(\alpha, K_b d)$ can be obtained.

Using these curves, from the measured α and $k_b d$ values the corresponding average plasma density is determined.

Following the above discussed procedure the α versus $k_p d$ curves for a constant a (fig. 5) and for a constant $k_b d$ (fig. 6) have been computed, assuming for the diameter ratio b the experimental value 1.47. The curves are those corresponding to the lowest x_0 solution of the dispersion equation. In fact we believe that our experimental data refer to this solution, because the fields outside the plasma are in this case much larger than those of the higher order solutions.

Technically we had to face the problem of insuring a well defined standing wave pattern by using good reflective terminations at the discharge ends. The best results were obtained with a mesh of tungsten wires placed transversally inside the discharge tube together with an external metallic iris.

Using this type of highly reflective terminations, measurements have been performed using several frequencies in the 1 - 2 KMc/s range. To characterize discharge conditions the RF voltage applied to the plasma between the electrodes was monitored. Typical results are as follows : RF discharge voltage 600 V peak, $k_b d = 5.86$, $\alpha = 1.6$ and then $n = 3.4 \cdot 10^{11}$ elec./cm^3 ; RF discharge voltage 1500 V peak, $k_b d = 5.86$, $\alpha = 1.3$ and then $n = 5.5 \cdot 10^{11}$ elec./cm^3.

GROUP-VELOCITY MEASUREMENTS

Wavelength measurements using a sliding antenna probe have provided information only on the first mode propagation. To obtain experimentally Brillouin diagrams for the other modes a different approach must be used. For this purpose, instead of measuring the phase constant β versus the signal frequency, we measure the derivative $d\omega / d\beta$, that is the group velocity.

This velocity is determined from measurements of the transmission time delay using free space propagation as reference. The signal is detected by the same probe used for the transmission experiments, so that we are certainly measuring the properties of those signals, which propagate through the main plasma body (second and higher order modes).

For measuring the delay the 5 k Mc/s signal is amplitude modulated from a lower frequency sinusoidal signal (30 Mc/s) and the phase variations of this modulating signal at the receiver are measured. To achieve the derived accuracy, the modulating signal is converted in the receiver to a lower frequency (30 Kc/s), so that the delay, which is displayed on a CRT is larger than the propagation delay as the frequency ratio (10^3 in this case).

Group-velocity v_g is given by the formula :

$$v_g = \frac{c}{1 + 0.3 \frac{\Delta t}{l}} \tag{45}$$

where Δt (in μsec) is the variation of the displayed time delay passing from free space to plasma guided propagation and l (in meters) is the length of the changed path, which is set equal to the discharge tube length.

The block diagram is shown in fig. 7.

The electrical characteristics of each main block are as follows :

a) Microwave varactor modulator.- Its purpose is to amplitude modulate at 30 Mc the microwave signal. The index of modulation is 1, up to 100 mW of microwave power, and reduces to 0,5 at 1 W.
The levels of sideband harmonics are always at least 25 db lower than the 30 Mc modulation sideband.

b) 30 Mc Amplif. and A.G.C. - This is a 120 db gain, 0,5 Mc band-width, 30 Mc center frequency amplifier; the output is held constant within 10 ± 0,1 Volt over 90 db signal input variations.

c) 30 Mc quarts oscillator and 30 Mc + 30 Kc quartz L.O. - The two quartz oscillators are built in the same thermostatically controlled container, so that the 30 Kc beating frequency can be held constant within 3‰ peak to peak frequency variation over long time intervals.

The method and the equipment have been satisfactorily tested measuring the propagation time delays in coaxial cables of known lengths and in free space varying the path lengths. Δt maximum errors are of the order of .1 μsec, that is sufficient for our experiment.

The plane spiral antenna, placed inside the plasma, which was chosen for the wavelength measurements, was no longer used, because of its extremely short life in the presence of strong discharges. A plane disk was used in some of the earlier tests, but it was also abandoned because its design was found to be critical, the coupling characteristics being different for each tube we have constructed.

Then a new design was chosen for both the transmitting and the receiving antennas. It consists of a dielectric antenna, incorporated into the end plates of the discharge tube (see fig.8). These new antennas not only show more constant coupling characteristics, but, due to the

absence of metallic parts inside the discharge, allow to attain very good vacuum, to maintain gas purity and to avoid the presence of sputtered surfaces.

Using these antennas, however, the transmitted signals have always been at low level. The detected power was found to increase with the magnetic field up to the maximum attainable field ($\sim$ 3000 Gauss). To improve the signal to noise ratios at the receiver, a new larger power supply for the solenoid was then built (50 V, 1000 A, so that fields around 4500 Gauss are attained). Larger signals were detected at these fields.

A large amount of data was taken using the curved and the straight solenoids, discharge powers from a few hundred watts up to 4 KW, neon gas at pressures in the millimeter range. Time delays from 1 to over 20 μsec have been measured. Inconsistencies of these data, however, have shown that operating conditions are not uniquely determined by our measured data : signal frequency, magnetic field intensity and RF voltage across the discharge.

This implies that gas and glass surface conditions are changing with time and from one experiment to another. Consequently density and its distribution are also changing. Previously discussed theories predict that these variations may largely influence propagation characteristics.

This problem deserves then more work both theoretical and experimental.

CONCLUSIONS

The main results obtained during the two years period, covered by the present report, have been discussed.

In the theory they are :

1) The basic theory has been completed, including the $\omega_l < \omega$ region and deriving Brillouin diagrams.
2) The relations with well-known aspects of the propagation in electron beams have been discussed and understood; these results can be usefully applied in microwave tube theory.
3) The differences between whistler characteristics for propagation in an uniform ionosphere and in a well-defined ionospheric duct have been reconsidered on more general grounds. These results are definitive for the case of uniform plasma ducts, but it seems interesting to examine from the same point of view the case of non-uniform ducts.
4) Preliminary theories on the propagation characteristics in non-uniform plasma columns have been worked out. In spite of the obvious analytical difficulties of this problem, more work along these lines is highly recommended.

Experiments have provided the following results :

1) Previous interpretation of transmission experiments has been confirmed by further data, which cover a wider range of experimental conditions.
2) Propagation wavelength measurements have been successfully performed, but they concern only first mode propagation.
3) A satisfactory method for electron density measurements has been found. This method is based on wavelength measurements in a circular waveguide, beyond cut-off, which contains the plasma column concentrically around the axis.

4) Group velocity measurements have been performed by measuring the propagation time delays of the microwave signal in the plasma.

5) Operating conditions are not uniquely determined by the assumed fundamental parameters : signal frequency, magnetic intensity and RF voltage across the discharge. There are reasons to believe that gas and glass surface conditions are changing with time and from experiment to experiment. More work is required to insure known and repeatable conditions in the future experiments.

These conclusions indicate also some of the basic work which next year research program will have to accomplish. In particular we mention the further development of non-uniform plasma theory and the quantitative analysis of experiments performed under known and repeatable conditions.

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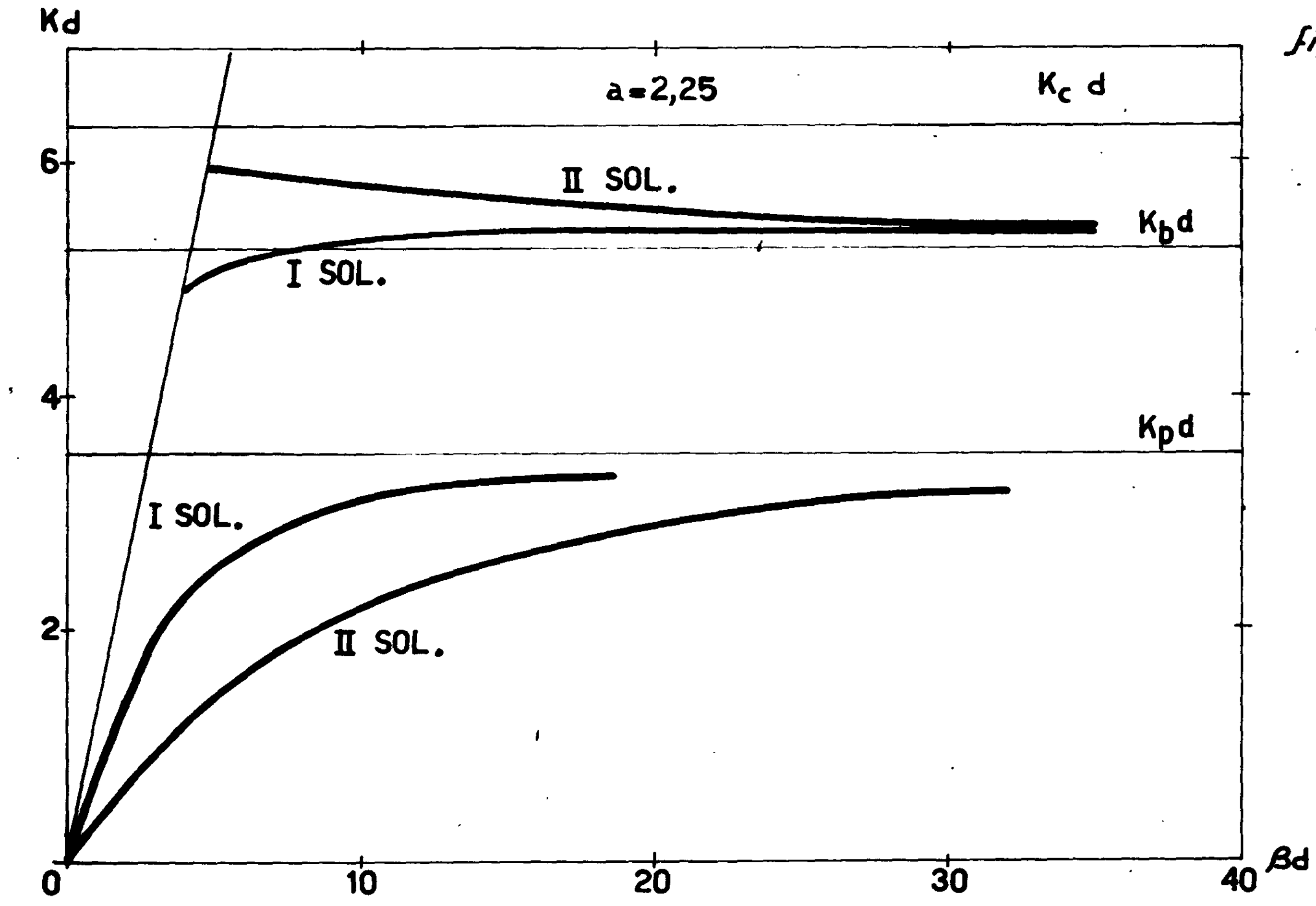
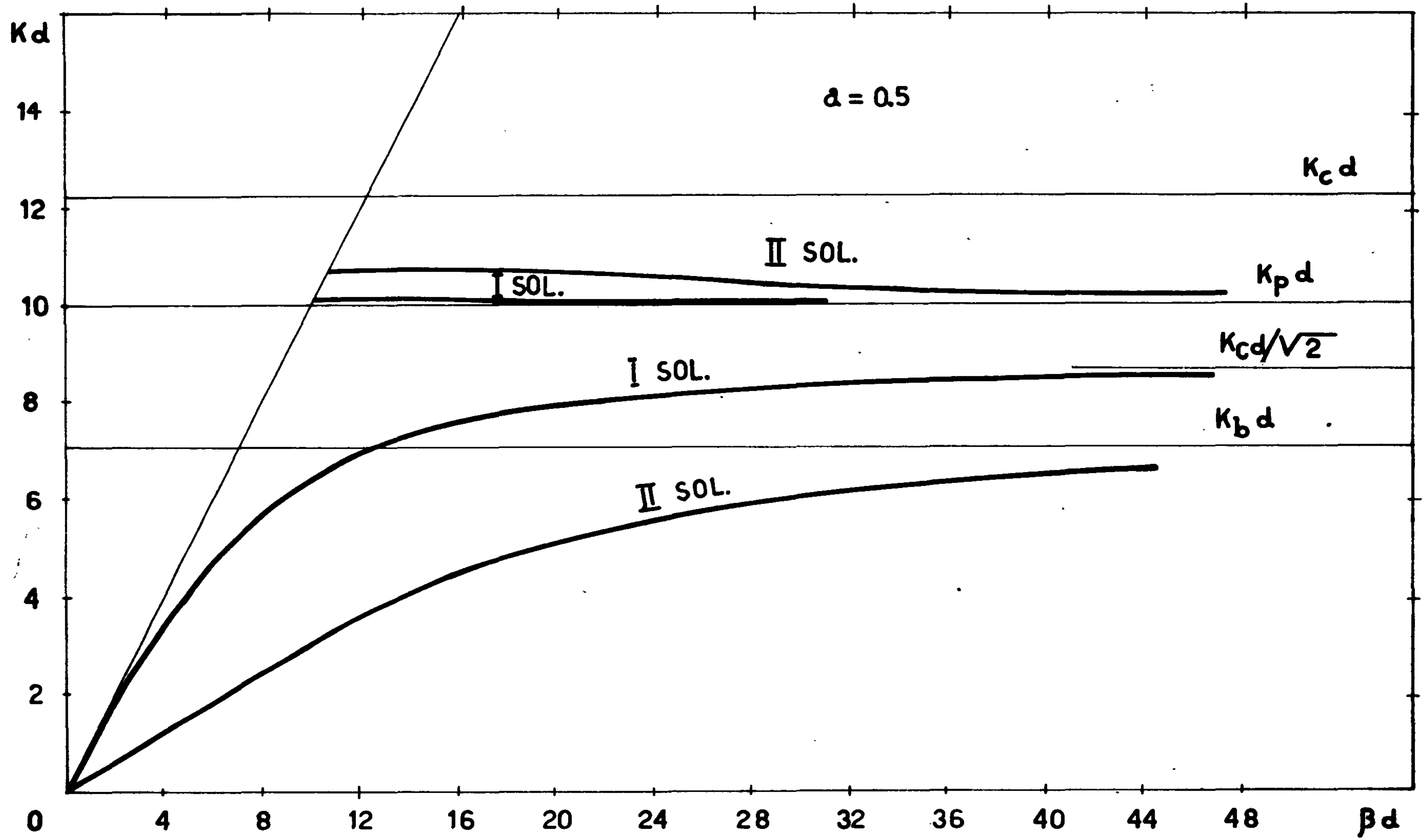


fig. 3


Kd
14
12
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βd
α = 0.5
$K_c d$
II SOL.
I SOL.
$K_p d$
$K_c d/\sqrt{2}$
I SOL.
$K_b d$
II SOL.
fig. 4

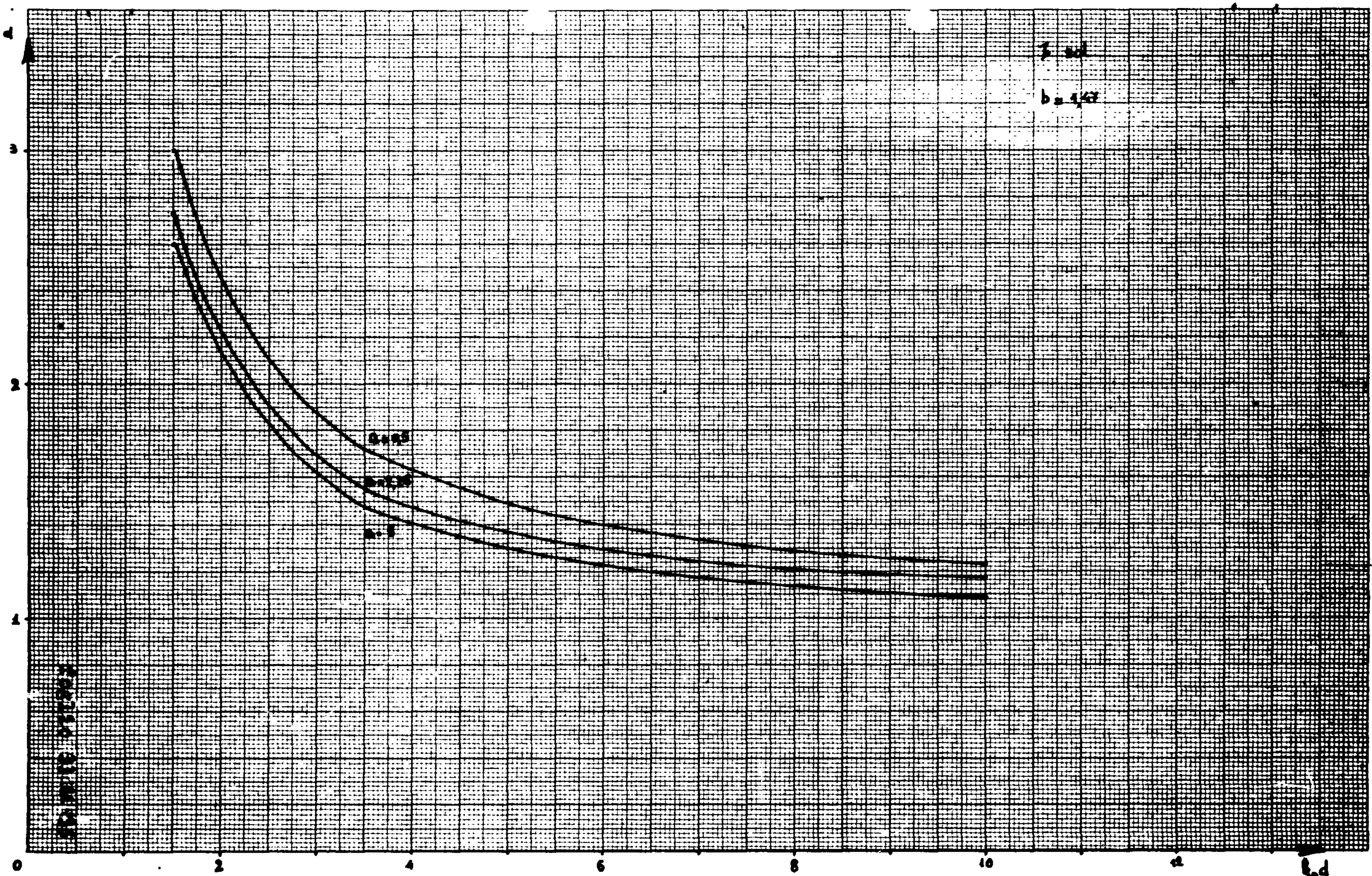

fig. 5

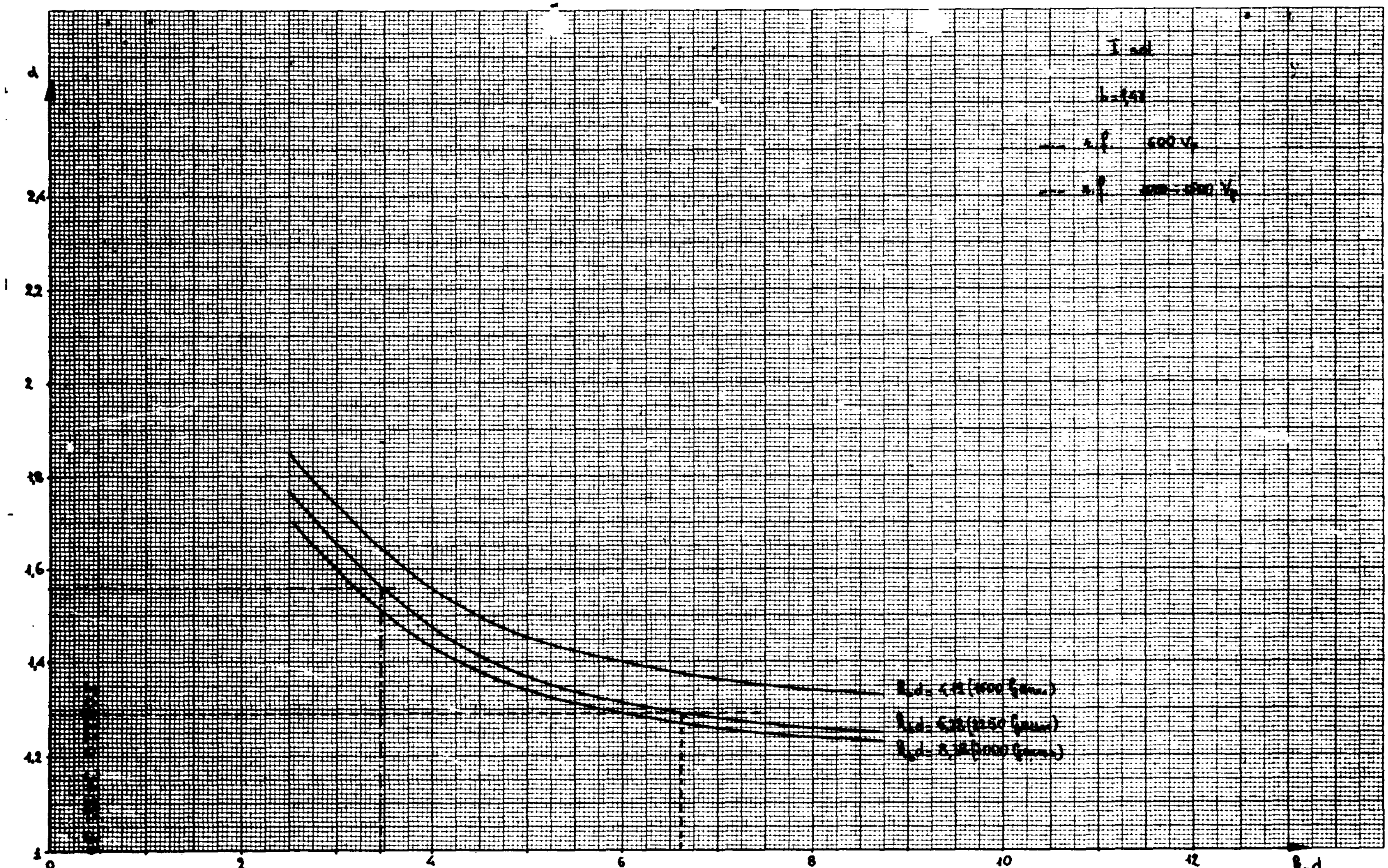

fig. 6

[redacted] S.p.A.

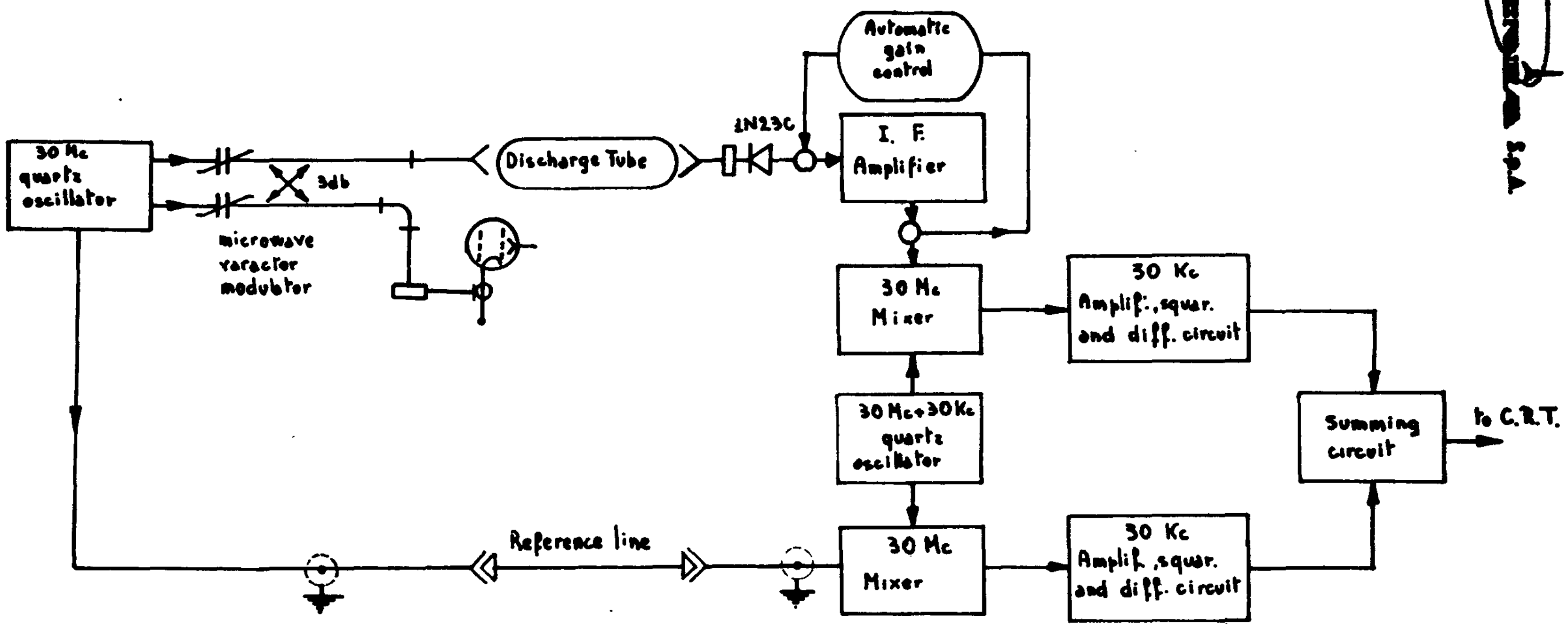

fig. 7

[redacted] S.p.A.

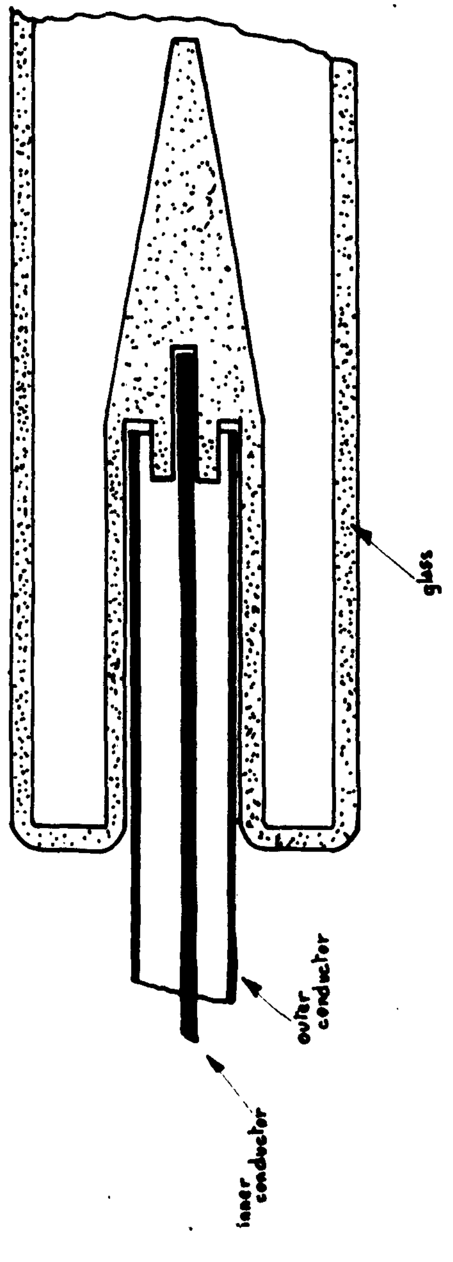

fig. 8

PROPAGATION IN INHOMOGENEOUS PLASMAS
(By Prof. Cambi 1960)

1- Introductory

The presence of free charges in a plasma can notoriously be taken into account by regarding the medium as possessing a tensorial permittivity as defined, in cartesian coordinates for instance, by

$$D_x = \varepsilon_0 \left[\varepsilon_1 E_x + j\varepsilon_2 E_y\right]$$

$$D_y = \varepsilon_0 \left[-j\varepsilon_2 E_x + \varepsilon_1 E_y\right]$$

$$D_z = \varepsilon_0 \varepsilon_3 E_z$$

where ε_0 is the absolute dielectric constant of free space and ε_1, ε_2, ε_3 are relative tensorial components defined by

$$\varepsilon_1 = 1 + \frac{\omega_p^2}{\omega_b^2 - \omega^2} \quad , \quad \varepsilon_2 = \frac{\omega_b}{\omega} \frac{\omega_p^2}{\omega_b^2 - \omega^2} \, , \quad \varepsilon_3 = 1 - \frac{\omega_p^2}{\omega^2} \qquad (1)$$

In these formulas, ω_p^2 is the square of the "plasma frequency" related to the electron density n by

$$\omega_p^2 = \frac{n e^2}{m \varepsilon_0}$$

while ω_b is the cyclotron frequency, related to the magnetic field by

$$\omega_b = eB/m .$$

In an inhomogeneous plasma duct, the electron density and/or the magnetic intensity are, generally speaking, functions of the point: in terms of the quantities

$$\delta = \frac{\omega_p}{\omega} \quad , \quad k = \frac{\omega_b}{\omega} \qquad (2)$$

the components of the permittivity tensor become

$$\varepsilon_1 = 1 + \frac{\delta^2}{k^2 - 1} \quad ; \quad \varepsilon_2 = \frac{k\,\delta^2}{k^2 - 1} \quad ; \quad \varepsilon_3 = 1 - \delta^2 \qquad (3)$$

If the coordinates are measured in terms of $\frac{\lambda_0}{2\pi} = \frac{c}{\omega} = 1/\omega\sqrt{\mu\varepsilon_0}$, that is, if $x = \frac{2\pi}{\lambda_0}$ times the physical abscissa, and so on; and if, further, $\vec{K}$ is written for the vector $Z_0\vec{H} = \sqrt{\mu/\varepsilon_0}\cdot\vec{H}$ (homogeneous with $\vec{E}$), the adimensional Maxwell equations become

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j K_x \qquad \frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z} = j(\varepsilon_1 E_x + j\varepsilon_2 E_y)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j K_y \qquad \frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x} = j(-j\varepsilon_2 E_x + \varepsilon_1 E_y) \qquad (4)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j K_z \qquad \frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y} = j\varepsilon_3 E_z .$$

As stated above, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are functions of the point through the quantities δ and k appearing in (3): but it is quite obvious that the analytical problem cannot be solved in the present generality. Actually, the problem is quite complicated even in the most schematic assumptions: so that it is necessary to discuss separately the most elementary laws of variability.

2- <u>TEM mode in a uniform plasma</u>

The simplest conceivable case is that of a plasma where the electron density, or the magnetic induction, or both, are functions of a single coordinate, say z. In this assumption, a TEM mode independent of x and y is obviously possible, satisfying the reduced system

$$-\frac{\partial E_y}{\partial z} = -j K_x \qquad -\frac{\partial K_y}{\partial z} = j\varepsilon_1 E_x - \varepsilon_2 E_y$$

$$\frac{\partial E_x}{\partial z} = -j K_y \qquad \frac{\partial K_x}{\partial z} = \varepsilon_2 E_x + j\varepsilon_1 E_y \qquad (5)$$

$$0 = K_z \qquad 0 = E_z .$$

ε_1 and ε_2 are to be regarded as known functions of z. Eliminating K_x and K_y, the system

$$\frac{d^2 E_y}{dz^2} + \varepsilon_1 E_y = j\varepsilon_2 E_x$$

$$\frac{d^2 E_x}{dz^2} + \varepsilon_1 E_x = -j\varepsilon_2 E_y \qquad (6)$$

is written at once.

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In spite of the apparent simplicity, the system is exceedingly complicated, even for the simplest laws of variability of ε_1, and ε_2. In view of this, a rather extended discussion is in order.

In a first instance, we consider the apparently trivial (but still delicate enough) case where ε_1 and ε_2 are constant, that is, the case of a TEM propagation in a uniform plasma. When ε_1 and ε_2 are constant, system (6) is solved by making E_x and E_y of the respective forms $e_1 e^{j\mu z}$, $e_2 e^{j\mu z}$ provided μ is a root of

$$(\varepsilon_1 - \mu^2)^2 - \varepsilon_2^2 = 0.$$

The possible exponents are, accordingly

$$\pm\alpha = \pm\sqrt{\varepsilon_1 - \varepsilon_2} \quad ; \quad \pm\beta = \pm\sqrt{\varepsilon_1 + \varepsilon_2}.$$

From the general expression of E_x

$$E_x = k_1 e^{j\alpha z} + k_2 e^{-j\alpha z} + k_3 e^{j\beta z} + k_4 e^{-j\beta z}$$

that of E_y is found at once to be

$$E_y = jk_1 e^{j\alpha z} + jk_2 e^{-j\alpha z} - jk_3 e^{j\beta z} - jk_4 e^{-j\beta z}.$$

The first Maxwell equations also give

$$K_x = j\alpha k_1 e^{j\alpha z} - j\alpha k_2 e^{-j\alpha z} - j\beta k_3 e^{j\beta z} + j\beta k_4 e^{-j\beta z}$$

$$K_y = -\alpha k_1 e^{j\alpha z} + \alpha k_2 e^{-j\alpha z} - \beta k_3 e^{j\beta z} + \beta k_4 e^{-j\beta z}.$$

If α and β are taken positive (assuming that $\varepsilon_1 > \varepsilon_2$) the terms with negative exponents represent a forward propagation: assuming that the field originates "from the left", and that no discontinuities occur, these are the only components of interest.

Of course, in order to excite, say at $z=0$, such a progressive field, the initial $\vec{E}$ and $\vec{K}$ should match the local characteristic impedance; that is, they are not simultaneously arbitrary.

Taking as x- direction that of $\vec{E}$ at $z=0$ and denoting by E the (constant) modulus, a progressive field is obtained by making $k_1 = k_3 = 0$, $k_2 = k_4 = \frac{E}{2}$: the propagating field is then

$$E_x = \frac{E}{2}\left\{e^{-j\alpha z} + e^{-j\beta z}\right\} \quad ; \quad K_x = -j\frac{E}{2}\left\{\alpha e^{-j\alpha z} - \beta e^{-j\beta z}\right\}$$

$$E_y = j\frac{E}{2}\left\{e^{-j\alpha z} - e^{-j\beta z}\right\} \quad ; \quad K_y = \frac{E}{2}\left\{\alpha e^{-j\alpha z} + \beta e^{-j\beta z}\right\}.$$

This can be regarded as resulting from the superposition of an "alfa field"

$$E'_x = \frac{E}{2} e^{-j\alpha z} \qquad ; \qquad K'_x = -j\frac{E}{2}\alpha e^{-j\alpha z}$$
$$E'_y = j\frac{E}{2} e^{-j\alpha z} \qquad ; \qquad K'_y = \frac{E}{2}\alpha e^{-j\alpha z} \qquad ;$$

with a "beta field" similarly defined. The velocity of propagation of the α-field is c/α, that of the beta field c/β.

In each of the two fields the vector $\vec{K}$ is perpendicular to $\vec{E}$, but the Poynting vector is zero: in other words, the fields, isolated, do not convey any power. Power propagates by virtue of the superposition of the two fields: actually

$$\vec{E}' \times \vec{K}'' = \frac{E^2}{2}\beta e^{-j(\alpha+\beta)z}$$
$$\vec{E}'' \times \vec{K}' = \frac{E^2}{2}\alpha e^{-j(\alpha+\beta)z}.$$

In the alfa field, the plane of polarization rotates negatively: actually, denoting by φ the angle of $\vec{E}$ with x, we have

$$\tan\varphi' = \frac{E'_y}{E'_x} = -\tan(\omega t - \alpha z)$$

while in the beta field

$$\tan\varphi'' = \frac{E''_y}{E''_x} = \tan(\omega t - \beta z).$$

When the two fields are superimposed the polarization is fixed with time, but depends on z: that is, the polarization continuously rotates in the course of the propagation.
Actually:

$$\tan\varphi(z) = \frac{E'_y + E''_y}{E'_x + E''_x} = \frac{\sin(\omega t - \beta z) - \sin(\omega t - \alpha z)}{\cos(\omega t - \beta z) + \cos(\omega t - \alpha z)} = -\tan\frac{\alpha-\beta}{2} z.$$

3- TEM modes in a longitudinally variable plasma

The more general system (6) where ε_1 and ε_2 are generic functions of z alone can be attacked in substantially the same way.
Looking for a function F of z such that $E_x = C_1 F$ and $E_y = C_2 F$

may satisfy the system, we find the conditions

$$(\ddot{F} + \varepsilon_1 F)C_1 + j\varepsilon_2 F C_2 = 0$$

$$-j\varepsilon_2 F C_1 + (\ddot{F} + \varepsilon_1 F)C_2 = 0.$$

Non zero solution C_1 and C_2 exist if, and only if

$$\ddot{F} + (\varepsilon_1 \mp \varepsilon_2)F = 0. \qquad (7)$$

In spite of the apparent elementarity, the equation is exceedingly complicated unless ε_1 and ε_2 are constant (in which case the classic solution is found). To make the problem practical, we shall accept first the reasonable assumption that the variation of ε_1 and ε_2 with z be relatively slow (as referred to the wavelength) so that in a substantial number of "wavelengths" the variation may be regarded as linear. In this case the equation becomes of the form

$$\ddot{F} + (A + Bz)F = 0 \qquad (8)$$

and has the complicated solutions

$$F = \sqrt{A + Bz}\, J_{\pm\frac{1}{3}}\left[\frac{2}{3B}(A + Bz)^{3/2}\right] \qquad (9)$$

where J is the Bessel function.

In the present case A + Bz is, alternatively, $\sqrt{\varepsilon_1 - \varepsilon_2}$ or $\sqrt{\varepsilon_1 + \varepsilon_2}$; so that, extending the value of the definitions $\alpha = \sqrt{\varepsilon_1 - \varepsilon_2}$ and $\beta = \sqrt{\varepsilon_1 + \varepsilon_2}$ to the present case where α and β are functions of z, the general expression of, say, E_x is

$$E_x = \alpha\left\{C_1 J_{\frac{1}{3}}\left(\frac{2\alpha^3}{3b_1}\right) + C_2 J_{-\frac{1}{3}}\left(\frac{2\alpha^3}{3b_1}\right)\right\} + \beta\left\{C_3 J_{\frac{1}{3}}\left(\frac{2\beta^3}{3b_2}\right) + C_4 J_{-\frac{1}{3}}\left(\frac{2\beta^3}{3b_2}\right)\right\}$$

with $b_1 = \dot{\varepsilon}_1 - \dot{\varepsilon}_2$, $b_2 = \dot{\varepsilon}_1 + \dot{\varepsilon}_2$. (The formula, of course, is only valid when ε_1 and ε_2 are <u>linear</u> functions of z). When F satisfies (7) with the upper (minus) sign, the coefficients of E_y are + j times those of E_x; the converse is true in the case of the lower sign.

Accordingly

$$E_y = j\alpha\left\{C_1 J_{\frac{1}{3}}\left(\frac{2\alpha^3}{3b_1}\right) + C_2 J_{-\frac{1}{3}}\left(\frac{2\alpha^3}{3b_1}\right)\right\} - j\beta\left\{C_3 J_{\frac{1}{3}}\left(\frac{2\beta^3}{3b_2}\right) + C_4 J_{-\frac{1}{3}}\left(\frac{2\beta^3}{3b_2}\right)\right\}.$$

It is seen that the knowledge of a closed solution of equation (7) when $t_1 \pm t_2$ is linear is far from constituting an advantage: in addition to being complicated, solutions of form (9) are not physically evident: for instance, the way is by no means clear by which combinations of functions (9) may tend to $e^{\pm j\alpha z}$ or $e^{\pm j\beta z}$, as is necessary, when α and β tend to become constant (b_1 and b_2 to zero)

The remark suggests the opportunity of looking for a more practical solution of the physical problem as represented by the general equation

$$\frac{\ddot{F}}{F} = -g(z)$$

where $g(z) = t_1 \pm t_2$ is a known function of z. The assumption of a linear behavior of g is only an approximation which obviously looses interest if it does not lead to a simplification. Writing $F = e^{\varphi}$, as is natural to do, the equation transforms into a Riccati equation

$$\dot{\varphi}^2 + \ddot{\varphi} = -g(z) \qquad (10)$$

The fact that g is normally negative, and real, suggests for $\dot{\varphi}$ a complex form $\dot{\varphi} = \dot{a} + j\dot{b}$: this leads to

$$\ddot{a} + j\ddot{b} + \dot{a}^2 - \dot{b}^2 + 2j\dot{a}\dot{b} = -g(z).$$

For the reality of the lefthand side, $\ddot{b} + 2\dot{a}\dot{b}$ must vanish: this gives $\dot{b} = he^{-2a}$ and

$$\ddot{a} + \dot{a}^2 - h^2 e^{-4a} = -g(z).$$

The further position a = ln u yields

$$\frac{\ddot{u}}{u} - \frac{h^2}{u^4} = -g(z) \qquad (11)$$

The physical problem is thus reduced to that of assuming a convenient form of u which, when substituted into (11), may yield a reasonable approximation to g(z). This can be done in many different ways. For example, a function $u = e^{\gamma z}$, that is, $a = \gamma z$, could approximate g(z) by the expression

$$h^2 e^{-4\gamma z} - \gamma^2,$$

the parameters of which can be so chosen as to reasonably "match" the actual $g(z)$. For a match near the origin, for instance, the conditions

$$h^2 - \gamma^2 = g(0) \qquad ; \qquad -4\gamma h^2 = g'(0)$$

are written.

The corresponding $\dot{b}$ is $h e^{-2\gamma z}$: as the components of the field have been assumed to be of the form e^{a+jb} the "instantaneous wavelength" is $2\pi/\dot{b} = (2\pi/h)\, e^{2\gamma z}$.
From the initial value $2\pi/h$ the wavelength increases if γ is positive: in this case the amplitude also increases.

The complete expression is

$$a + jb = \gamma z + jh \int e^{-2\gamma z}\, dz = \gamma z + j\frac{h}{2\gamma}\left[1 - e^{-2\gamma z}\right]$$

that is, neglecting a multiplicative constant

$$F = \exp\left[\gamma z - \frac{jh}{2\gamma}\, e^{-2\gamma z}\right]$$

For given $g(0)$ and $g'(0)$, is the real root of the equation

$$4\gamma^3 + 4\, g(0)\gamma + g'(0) = 0.$$

As h is real by assumption, γ is negative or positive according as $g'(0)$ is positive or negative.

According to (3), the two physical expressions for the function g are

$$\varepsilon_1 + \varepsilon_2 = 1 + \frac{\delta^2}{k-1} \qquad , \qquad \varepsilon_1 - \varepsilon_2 = 1 - \frac{\delta^2}{k+1} \quad ,$$

where $k = \omega_b/\omega$ is proportional to the static magnetic induction and $\delta^2 = e^2 n / m \varepsilon_0 \omega^2$ is proportional to the electron density. If the variation of g (referred to the wavelength) is slow, γ is small and h^2 is close to $g(0)$, that is, to α^2 or β^2.

Given the four constants $h_1, h_2, \gamma_1, \gamma_2$ as defined by the

initial values of $\varepsilon_1, \varepsilon_2$ and their derivatives we thus have

$$E_x = C_1 \exp\left(\gamma_1 z - \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + C_2 \exp\left(\gamma_1 z + \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + C_3 \exp\left(\gamma_2 z - \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right) + C_4 \exp\left(\gamma_2 z + \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right)$$

$$E_y = jC_1 \exp\left(\gamma_1 z - \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + jC_2 \exp\left(\gamma_1 z + \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) - jC_3 \exp\left(\gamma_2 z - j\frac{h_2}{2\gamma_2} e^{-2\gamma_2 z}\right) - jC_4 \exp\left(\gamma_2 z + \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right)$$

if the constants h_1, γ_1 are related to $\varepsilon_1 - \varepsilon_2$ and h_2, γ_2 to $\varepsilon_1 + \varepsilon_2$.

The Maxwell equations give

$$\begin{aligned} K_x = {} & C_1\left(\gamma_1 + jh_1 e^{-2\gamma_1 z}\right) \exp\left(\gamma_1 z - \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + \\ & + C_2\left(\gamma_1 - jh_1 e^{-2\gamma_1 z}\right) \exp\left(\gamma_1 z + \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) - \\ & - C_3\left(\gamma_2 + jh_2 e^{-2\gamma_2 z}\right) \exp\left(\gamma_2 z - \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right) - \\ & - C_4\left(\gamma_2 - jh_2 e^{-2\gamma_2 z}\right) \exp\left(\gamma_2 z + \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right); \end{aligned}$$

$$\begin{aligned} K_y = {} & jC_1\left(\gamma_1 + jh_1 e^{-2\gamma_1 z}\right) \exp\left(\gamma_1 z - \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + \\ & + jC_2\left(\gamma_1 - jh_1 e^{-2\gamma_1 z}\right) \exp\left(\gamma_1 z + \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}\right) + \\ & + jC_3\left(\gamma_2 + jh_2 e^{-2\gamma_2 z}\right) \exp\left(\gamma_2 z - \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right) + \\ & + jC_4\left(\gamma_2 - jh_2 e^{-2\gamma_2 z}\right) \exp\left(\gamma_2 z + \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}\right). \end{aligned}$$

It is easily recognized that if h_1 and h_2 are chosen positive, the progressive fields are those involving + j in the exponentials: assuming that the entering field at z = 0 is matched to the local impedance the conditions for the progressive field where $E_x(0) = E$, $E_y(0) = 0$ are

$$C_2\, e^{jh_1/2\gamma_1} + C_4\, e^{jh_2/2\gamma_2} = E$$

$$C_2\, e^{jh_1/2\gamma_1} - C_4\, e^{jh_2/2\gamma_2} = 0$$

$$C_1 = C_3 = 0 .$$

Accordingly, and writing for brevity

$$e^{\gamma_1 z + \frac{jh_1}{2\gamma_1} e^{-2\gamma_1 z}} = \exp_1 \quad ; \quad e^{\gamma_2 z + \frac{jh_2}{2\gamma_2} e^{-2\gamma_2 z}} = \exp_2 ,$$

the field is

$$E_x = \frac{E}{2} e^{-\frac{jh_1}{2\gamma_1}} \exp_1 + \frac{E}{2} e^{-\frac{jh_2}{2\gamma_2}} \exp_2$$

$$E_y = j\frac{E}{2} e^{-\frac{jh_1}{2\gamma_1}} \exp_1 - j\frac{E}{2} e^{-\frac{jh_2}{2\gamma_2}} \exp_2$$

$$K_x = \frac{E}{2}\left(\gamma_1 - jh_1 e^{-2\gamma_1 z}\right) e^{-\frac{jh_1}{2\gamma_1}} \exp_1 - \frac{E}{2}\left(\gamma_2 - jh_2 e^{-2\gamma_2 z}\right) e^{-\frac{jh_2}{2\gamma_2}} \exp_2$$

$$K_y = j\frac{E}{2}\left(\gamma_1 - jh_1 e^{-2\gamma_1 z}\right) e^{-\frac{jh_1}{2\gamma_1}} \exp_1 + j\frac{E}{2}\left(\gamma_2 - jh_2 e^{-2\gamma_2 z}\right) e^{-\frac{jh_2}{2\gamma_2}} \exp_2 .$$

In every expression, the first term represents the "alfa field", the other the "beta field". As usual, the Poynting vector pertaining to each separate field vanishes identically, while

$$\vec{E}' \times \vec{K}'' = j\frac{E^2}{2} e^{-\frac{jh_1}{2\gamma_1} - \frac{jh_2}{2\gamma_2}} \left(\gamma_2 - jh_2 e^{-2\gamma_2 z}\right) \exp_1 \exp_2 \; ; \quad \text{etc.}$$

The discussion of the field: rotation, attenuation, etc. is carried out without difficulty: but the consideration of a more concrete case is of greater interest.

4- Propagation in a longitudinally variable laminar plasma duct

As a next case, in order of increasing complication, we can consider that of a plasma distribution that, in a cartesian space, may show a variation in the z-direction alone, as above, but is confined in a finite width in the x-direction being unlimited, however, along the y-axis. If we look for fields which are independent of y, the Maxwell equations become

$$-\frac{\partial E_y}{\partial z} = -jK_x \qquad -\frac{\partial K_y}{\partial z} = j\varepsilon_1 E_x - \varepsilon_2 E_y$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -jK_y \qquad \frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x} = \varepsilon_2 E_x + j\varepsilon_1 E_y$$

$$\frac{\partial E_y}{\partial x} = -jK_z \qquad \frac{\partial K_y}{\partial x} = j\varepsilon_3 E_z$$

that is, eliminating $\vec{K}$,

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_z}{\partial z \partial x} = -\varepsilon_1 E_x - j\varepsilon_2 E_y$$

$$\frac{\partial^2 E_y}{\partial z^2} + \frac{\partial^2 E_y}{\partial x^2} = j\varepsilon_2 E_x - \varepsilon_1 E_y \qquad (12)$$

$$\frac{\partial^2 E_x}{\partial z \partial x} - \frac{\partial^2 E_z}{\partial x^2} = \varepsilon_3 E_z .$$

It is recalled that ε_1 ε_2 ε_3 are functions of z alone.

The question is spontaneous whether a solution can exist of he form $\vec{E} = \vec{A}F$ where the vector $\vec{A}$ is function of x and the scalar F of z only. Writing for brevity $\dot{F}/F = \dot{\varphi}$ as before, and $\ddot{F}/F = -g = -\dot{\varphi}^2 - \ddot{\varphi}$, the system becomes

$$-A_x(g-\varepsilon_1) + j\varepsilon_2 A_y = A'_z \dot{\varphi}$$

$$A''_y - A_y(g-\varepsilon_1) - j\varepsilon_2 A_x = 0 \qquad (13)$$

$$A'_x \dot{\varphi} = A''_z + \varepsilon_3 A_z$$

(Accents denote differentiation in respect to x, dots in respect to z).

System (13), where the coefficients are functions of z, should admit of solutions A_x, A_y, A_z, functions of x only. The requirement is very restrictive, and for given ε_1 ε_2 ε_3 cannot be satisfied by a single function $\dot{\varphi}$. This means that the existence of a solution of type $\vec{AF}$ is an exceptional event, if even possible.

The investigation whether of not such a plane propagation can take place is relatively simple. If the first two equations (13) are written

$$A_x \frac{g-\varepsilon_1}{\dot{\varphi}} + A_y \frac{j\varepsilon_2}{\dot{\varphi}} = A'_z$$

$$j\varepsilon_2 A_x + A_y (g-\varepsilon_1) = A''_y$$

we find, upon differentiation in respect to z:

$$-\frac{d}{dz}\frac{g-\varepsilon_1}{\dot{\varphi}} \cdot A_x + j\frac{d}{dz}\frac{\varepsilon_2}{\dot{\varphi}} \cdot A_y = 0$$

$$j\frac{d\varepsilon_2}{dz} \cdot A_x + \frac{d(g-\varepsilon_1)}{dz} \cdot A_y = 0$$

this requires, first, that

$$\frac{d}{dz}(g-\varepsilon_1) \cdot \frac{d}{dz}\frac{g-\varepsilon_1}{\dot{\varphi}} = \frac{d}{dz}\frac{\varepsilon_2}{\dot{\varphi}} \cdot \frac{d}{dz}\varepsilon_2$$

Furthermore, the ratio

$$\frac{d(g-\varepsilon_1)/dz}{d\varepsilon_2/dz}$$

must be independent of z: that is, as ε_1 and ε_2 depend on z only, must be a constant. This means

$$g-\varepsilon_1 = C_1\varepsilon_2 + C_2 \qquad (14)$$

Replacing in the first condition, we find

$$(C_1^2-1)\frac{d}{dz}\frac{\varepsilon_2}{\dot\varphi} + C_1C_2\frac{d}{dz}\frac{1}{\dot\varphi} = 0$$

and

$$A_x = j\,C_1 A_y .$$

The equations are necessary consequences of the assumption of a form $\vec{A}F(z)$ for the vector $\vec{E}$ (plane waves).

Replacing in the first two (13), we have

$$-j\left[(C_1^2-1)\varepsilon_2 + C_1C_2\right]A_y = A_z'\dot\varphi$$

$$A_y'' - C_2A_y = 0.$$

The second relation is perfectly compatible with the independence of A_y from z; the first equation requires that the ratio

$$\frac{(C_1^2-1)\varepsilon_2 + C_1C_2}{\dot\varphi} = D \qquad (15)$$

be a constant. ($\dot\varphi$ is connected to ε_1 and ε_2 by (14))

From $A_z' = -jDA_y$ we find, replacing in the third (13)

$$-jD(A_y'' + \varepsilon_3 A_y) = A_x''\dot{\varphi}$$

that is

$$-jD(C_2+\varepsilon_3)A_y = jC_1A_y''\dot{\varphi} = jC_1C_2A_y\dot{\varphi}$$

or finally

$$D(C_2+\varepsilon_3) + C_1C_2\dot{\varphi} = 0. \qquad (16)$$

(14),(15) and (16) represent the necessary conditions for the existence of plane waves: $\dot{\varphi}$ can readily be eliminated, writing from (15)

$$\dot{\varphi} = \frac{(C_1^2-1)\varepsilon_2 + C_1C_2}{D} = H\varepsilon_2 + K$$

that is, replacing into(14),

$$-(H\varepsilon_2+K)^2 - H\dot{\varepsilon}_2 = C_1\varepsilon_2 + \varepsilon_1 + C_2 \qquad (17)$$

and,replacing into(16)

$$D(C_2+\varepsilon_3) + C_1C_2[H\varepsilon_2+K] = 0. \qquad (18)$$

The conditions should be satisfied by suitable values of the constants C_1 C_2 and D (to which H and K are obviously connected). As ε_3 is expressible in terms of ε_1 and ε_2, the two equations, with given constants, completely determine the plasma configuration. It is possible that the conditions be less restrictive than they appear at first, owing to the presence of three arbitrary constants which could possibly match well enough the actual situation.

For a given configuration the problem only imposes two conditions upon the three constants, one of which could possibly be left free for a best possible match: if the configuration varies rather slowly with z, a solution by steps is then conceivable.

5-Suggested method for the solution of more general cartesian cases

Cases of wider generality than those that have been considered hitherto cannot be treated by methods of comparable simplicity. The description of a computational method that may presumably be valid in a larger variety of cases is thus in order. We assume, as before, that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be functions of z only, but their generality is not restricted at the moment.

The solutions of the Maxwell equations are supposed to be expanded in series of orthogonal functions of x, having functions of z as coefficients. If the width of the channel is indicated by 2w, $\cos\frac{n\pi}{w}x$ and $\sin\frac{n\pi}{w}x$ can constitute such a set of orthogonal functions. A simple inspection of equations (12) shows that E_x and E_y, as functions of x, have the same parity, while E_z is opposite: the same is true of the components of K. Accordingly, "even" harmonic components can be written as

$$E_x = e_x \cos\frac{n\pi}{w}x \qquad E_y = e_y \cos\frac{n\pi}{w}x \qquad E_z = e_z \sin\frac{n\pi}{w}x \tag{22}$$

"odd" components as

$$E_x = e_x \sin\frac{n\pi}{w}x \qquad E_y = e_y \sin\frac{n\pi}{w}x \qquad E_z = -e_z \cos\frac{n\pi}{w}x$$

Similar expression are valid for K_x, K_y, K_z in terms of harmonic components k_x, k_y, k_z. (These components are actually functions of the order n: the corresponding index, however, is omitted for simplicity.) With the notations accepted in (22), the harmonic components appear to be connected by the ordinary system:

$$\begin{aligned} \dot{e}_y &= j\,k_x & -\dot{k}_y &= j\varepsilon_1\, e_x - \varepsilon_2\, e_y \\ \dot{e}_x &= \frac{n\pi}{w} e_z - j\,k_y & \dot{k}_x &= \frac{n\pi}{w} k_z + \varepsilon_2\, e_x + j\varepsilon_1\, e_y \\ \frac{n\pi}{w} e_y &= j\,k_z & -\frac{n\pi}{w} k_y &= j\varepsilon_3\, e_z \end{aligned} \tag{23}$$

The equations are valid both for the even and for the odd components.

Even in the simplest assumptions as to the form of the functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of z, the resolution of (23) is exceedingly complicated. If attention is given to the fact that one system (23) exists for any positive value of n, the necessity of some standard method of solution, at least approximated, is obvious.

Regarding the variation of the parameters as a relatively small perturbation in respect to the constancy, we write $\varepsilon_1 = \varepsilon_{10} + z\zeta_1$ etc., where, of course, $\zeta_1 = \dot{\varepsilon}_{10} + \frac{\ddot{\varepsilon}_{10}}{2!} z + \frac{\dddot{\varepsilon}_{10}}{3!} z^2 \ldots$ Assuming that $\bar{e}_x, \ldots, \bar{k}_z$ satisfy the static system

$$\dot{\bar{e}}_y = j\bar{k}_x \qquad -\dot{\bar{k}}_y = j\varepsilon_{10}\bar{e}_x - \varepsilon_{20}\bar{e}_y$$

$$\dot{\bar{e}}_x = \frac{n\pi}{W}\bar{e}_z - j\bar{k}_y \qquad \dot{\bar{k}}_x = \frac{n\pi}{W}\bar{k}_z + \varepsilon_{20}\bar{e}_x + j\varepsilon_{10}\bar{e}_y \qquad (24)$$

$$\frac{n\pi}{W}\bar{e}_y = j\bar{k}_z \qquad -\frac{n\pi}{W}\bar{k}_y = j\varepsilon_{30}\bar{e}_z$$

we look for solutions of system (23) written in the form $\bar{e}_x + f_x, \ldots, \bar{k}_z + \ell_z$, where it is supposed that products of small terms of the form of f times $z\zeta$, are negligible in the whole useful range of z in comparison to the principal terms of form e times ε_0. It is readily found that the corrective terms f and ℓ have to satisfy the system

$$\dot{f}_y = j\ell_x \qquad -\dot{\ell}_y = j\zeta_1\bar{e}_x z - \zeta_2\bar{e}_y z + j\varepsilon_{10}f_x - \varepsilon_{20}f_y$$

$$\dot{f}_x = \frac{n\pi}{W}f_z - j\ell_y \qquad \dot{\ell}_x = \frac{n\pi}{W}\ell_z + \zeta_2\bar{e}_x z + j\zeta_1\bar{e}_y z + \varepsilon_{20}f_x + j\varepsilon_{10}f_y$$

$$\frac{n\pi}{W}f_y = j\ell_z \qquad -\frac{n\pi}{W}\ell_y = j\zeta_3\bar{e}_z z + j\varepsilon_{30}f_z \qquad (25)$$

If the functions $\bar{e}_x, \ldots, \bar{k}_z$, solutions of (23), have been so constructed as to satisfy the boundary conditions at z=o, the correcting functions f and ℓ can be subject to the simple condition of having zero initial values.

Application of the method first requires solution of system (24) where the coefficients are constant. (Of course, in the case n = 0, the field is TEM). The constant-coefficient system obviously admits of solutions of the form

$$e_x, e_y, \ldots, k_z = \text{constant} \times e^{\gamma z}$$

provided γ has such a value as to make zero the determinant of the algebraic system of the"constants".

Before entering in further detail, we remark that the equation for γ is biquadratic and that for every value of γ a solution of the form

$$\bar{e}_x = a_x e^{\gamma z}, \quad \bar{e}_y = a_y e^{\gamma z}, \ldots \quad \bar{k}_z = b_z e^{\gamma z}$$

is available, with constant $a_x, \ldots, b_z$ (one of them arbitrary)

System (25) is substantially of the same form as (24), but is complete instead of homogeneous, the "forcing terms" being of the form

$$j\zeta_1 \bar{e}_x z - \zeta_2 e_y z \; ; \; \zeta_2 \bar{e}_x z + j\zeta_1 \bar{e}_y z \; ; \; j\zeta_3 \bar{e}_z z$$

that is, of the form

$$\text{constant} \times z e^{\gamma z}$$

The analytical problem is thus solved as soon as one particular solution of the complete system is known. This can be done with standard rules as soon as "a fundamental system" of solutions of the homogeneous system is known. In the present case, the "fundamental system" is merely constituted by the set of four sextuples of functions corresponding to the four different values of γ.

In a somewhat more detailed discussion, we may observe, first, that the unknown functions are actually four rather than six, since two, out of equations (24), are in finite terms. Eliminating $\bar{e}_z$ and $\bar{k}_z$, (24) is replaced by the system of four equations:

$$\dot{\bar{e}}_y = j\,\bar{k}_x \qquad \dot{\bar{k}}_y = -j\,\varepsilon_{10}\,\bar{e}_x + \varepsilon_{30}\,\bar{e}_y$$
$$\dot{\bar{e}}_x = j\left(A^2/\varepsilon_{30} - 1\right)\bar{k}_y \qquad \dot{\bar{k}}_x = -j\left(A^2-1\right)\bar{e}_y + \varepsilon_{30}\,\bar{e}_x \tag{26}$$

where A is written for brevity in the place of $n\pi/W$. Similarly, system (25) becomes

$$\dot{f}_y = j\,\ell_x$$
$$\dot{f}_x = j\left(A^2/\varepsilon_{30} - 1\right)\ell_y - A\,\zeta_3\,z\,\bar{e}_z$$
$$\dot{\ell}_y = -j\,\varepsilon_{10}\,f_x + \varepsilon_{30}\,f_y + j\,\zeta_1\,z\,\bar{e}_x - \zeta_2\,z\,\bar{e}_y \tag{27}$$
$$\dot{\ell}_x = -j\left(A^2-1\right)f_y + \varepsilon_{30}\,f_x + \zeta_2\,z\,\bar{e}_x + j\,\zeta_1\,z\,\bar{e}_y$$

The solutions of (26) are of the form

$$\bar{e}_y = e^{\gamma_i z}, \quad \bar{e}_x = B_i\,e^{\gamma_i z}, \quad \bar{k}_y = C_i\,e^{\gamma_i z}, \quad \bar{k}_z = D_i\,e^{\gamma_i z}$$

$(i=1,2,3,4,)$ where γ_i is a root of the biquadratic equation obtained by making zero the determinant of the algebraic system obtained by replacing the last expression in (26), and B_i, C_i, D_i are the corresponding solutions. Writing for brevity

$$F_1 = 0, \quad F_2 = -A\zeta_3 z\,\bar{e}_z, \quad F_3 = j\zeta_1 z\,\bar{e}_x - \zeta_2 z\,\bar{e}_y, \quad F_4 = \zeta_2 z\,\bar{e}_x + j\zeta_1 z\,\bar{e}_y$$

the general solution of the complete system (27) is given by

$$f_y = e^{\gamma_1 z}\left[\int \Delta_1/\Delta\,dz + R_1\right] + e^{\gamma_2 z}\left[\int \Delta_2/\Delta\,dz + R_2\right] + e^{\gamma_3 z}\left[\int \Delta_3/\Delta\,dz + R_3\right] + e^{\gamma_4 z}\left[\int \Delta_4/\Delta\,dz + R_4\right]$$

$$f_x = B_1 e^{\gamma_1 z}\left[\int \Delta_1/\Delta\,dz + R_1\right] + B_2 e^{\gamma_2 z}\left[\int \Delta_2/\Delta\,dz + R_2\right] + B_3 e^{\gamma_3 z}\left[\int \Delta_3/\Delta\,dz + R_3\right] + B_4 e^{\gamma_4 z}\left[\int \Delta_4/\Delta\,dz + R_4\right] \tag{29}$$

ℓ_y = as above with coefficients C_1, C_2, C_3, C_4.

ℓ_x = as above with coefficients D_1, D_2, D_3, D_4

Δ is the determinant of the "fundamental system" namely

$$\Delta = \begin{vmatrix} e^{\gamma_1 z} & B_1 e^{\gamma_1 z} & C_1 e^{\gamma_1 z} & D_1 e^{\gamma_1 z} \\ e^{\gamma_2 z} & B_2 e^{\gamma_2 z} & C_2 e^{\gamma_2 z} & D_2 e^{\gamma_2 z} \\ e^{\gamma_3 z} & B_3 e^{\gamma_3 z} & C_3 e^{\gamma_3 z} & D_3 e^{\gamma_3 z} \\ e^{\gamma_4 z} & B_4 e^{\gamma_4 z} & C_4 e^{\gamma_4 z} & D_4 e^{\gamma_4 z} \end{vmatrix}$$

As $\gamma_1, \dots \gamma_4$ are solutions of a biquadratic equation, the sum $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is zero: so that Δ_1 reduces to the determinant of the constants. Similarly,

$$\Delta_1 = \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ e^{\gamma_2 z} & B_2 e^{\gamma_2 z} & C_2 e^{\gamma_2 z} & D_2 e^{\gamma_2 z} \\ e^{\gamma_3 z} & B_3 e^{\gamma_3 z} & C_3 e^{\gamma_3 z} & D_3 e^{\gamma_3 z} \\ e^{\gamma_4 z} & B_4 e^{\gamma_4 z} & C_4 e^{\gamma_4 z} & D_4 e^{\gamma_4 z} \end{vmatrix}$$

while $\Delta_2, \Delta_3, \Delta_4$ are similar in form but have F_1, F_2, F_3, F_4 respectively at the second, third, fourth row.

Functions F have obviously the form of a sum of terms $\zeta z\, e^{\gamma_i z}$ where ζ is a known function of z. The determinants Δ_i can thus be split in sums of four determinants the first of which has the first row proportional to $e^{\gamma_1 z}$, the second to $e^{\gamma_2 z}$ and so on. Out of these determinants the first is proportional to $e^{(\gamma_1+\gamma_2+\gamma_3+\gamma_4) z} = 1$ the second to $e^{(\gamma_2-\gamma_1) z}$ etc. In any case, the solutions (29) are explicit and can be directly evaluated, for any given set of functions $\zeta_1, \zeta_2, \zeta_3$

Of course, the solution is valid in the limit of the approximation consisting in regarding the pertubations as first order quantities. The functions $\bar{e}_x + f_x, \dots, \bar{K}_z + \ell_z$ which by virtue of the presence of the arbitrary constants $R_1, R_2 \dots, R_4$ have just the same forms (29) are solutions of a system of form (23) where, however, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are not the given functions of z but differ from them by terms of the order of the square of the perturbations.

The procedure can presumably be iterated, thus increasing the complication, but not the difficulty of the solution.

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ABSTRACT: The work performed under this Contract during the indicated two years period is decribed. Theoretical work includes : completion of the basic theory for propagation along plasma columns in magnetic fields, derivation of Brillouin diagrams, discussion of particularly significant limits and preliminary analyses of the non-uniform plasma case. Experimental work includes measurements of basic propagation parameters : transmitted signal, wavelength and group velocity, and of physical plasma parameters as the electron density.

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